Determination of the Hindered Settling Factor for Flocculated Suspensions

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The determination of the settling velocity of particles in suspension is a difficult problem due to the complexity of the indirect particle-particle interactions. Formally, a function called the hindered settling factor $r(\phi)$, which is a function of the solids volume fraction ϕ, is introduced to take into account the increase in the drag forces. The current method for estimating this function is to measure the initial fall rate of the suspension in a batch sedimentation experiment. Here, we propose a steady-state experiment for flocculated suspensions using vacuum filtration, where the fixed bed height is measured as a function of the liquid flow rate through a porous membrane. Recovering the function $r(\phi)$ or a related function $D(\phi)$ involves numerical differentiation of the data. This experiment will also yield the membrane permeability as a function of volume fraction (in cases where clogging or fouling of the membrane occurs). The results using numerically simulated data show good recovery of the assumed $r(\phi)$ function.

Introduction

A problem of widespread industrial and theoretical importance is the separation of fine solids from liquids. The suspended solids are consolidated under the influence of a body force applied to the particles, such as gravitational force in gravity thickening, an applied pressure in a pressure filter, or a pressure difference in a vacuum filter.

Each of these processes involves complex interactions among the particles. Even for the simplest problem with particles in a dilute suspension in a closed bottom container, the determination of the settling velocities of the particles is a hard problem due to indirect particle-particle interactions. This becomes much more difficult for concentrated and flocculated suspensions. Formally the effect of hydrodynamic interactions is taken into account by multiplying the drag coefficient by an interaction parameter, called the hindered settling factor $r(\phi)$, which is a function of the solids volume fraction ϕ. Clearly, $r(\phi)$ has the property $r(\phi)\rightarrow 1$ as $\phi\rightarrow 0$, and it is an increasing function of ϕ. Also, $r(\phi)\rightarrow\infty$ as $\phi\rightarrow 1$ or to a volume fraction corresponding to close packing ϕ_{cp}. The mean settling velocity is written as:

$$u=\frac{u_0(1-\phi)}{r(\phi)} \tag{1}$$

where u_0 is the velocity of a single particle in an unbounded medium (which depends on particle shape and must be determined). Batchelor (1971) calculated u for small values of ϕ as $u=(1-6.55\phi)$, which is the first term in an empirical relationship in the form $u=u_0(1-\phi)^a$. Such an empirical law, with $a\sim 4.5-6.5$, and others of the form $u=(1-\phi/\phi_{cp})^a$ and $u=(1-\phi)^{(a-b\phi)}$ have been fitted to experimental data, taken from measuring the initial settling rate or the rate of fall of the bed height for systems of varying concentration (Buscall et al., 1982; Kops-Werkhoven and Fijnaut, 1982; Auzerais et al., 1990). Since the settling velocities are difficult to measure, this article proposes an alternative method for determining the hindered settling factor $r(\phi)$ as a function of ϕ for flocculated suspensions. This arises from studying the mechanics of the dewatering process under the influence of a pressure difference in vacuum filter.

Buscall and White (1987) discussed the rheological properties of concentrated suspensions. The particles of the suspension interact directly with one another to give rise to a local particle pressure p_s, which is the effective stress tensor. When electrolyte or polymer flocculants have been added to the suspension, connected aggregate structures of many particles are produced and held together by van der Waals or polymer-bridging forces. Once the average particle volume fraction is high enough that a network of connected particles is formed, the suspension takes on the properties of a solid (albeit flimsy) structure. In

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particular, compressive stresses on the suspension can be transmitted via the network throughout the system, and then the structure possesses the ability to support itself. In a flocculated system above this volume fraction, the particle pressure p_s should be more properly thought of as a network pressure. When such a network has formed throughout the system, we are free to increase p_s by applying some sort of external compression to the network, for example, pushing on it with a piston or increasing the gravitational forces in a centrifuge. As this process is applied, the network structure will resist further compression, and p_s will increase until the compressive forces become so strong that the structure will begin to deform irreversibly. The rheological property to describe this is the compressive yield stress $P_y(\phi)$, which is defined as the value of the network pressure at which the flocculated suspension at volume fraction ϕ will no longer resist compression elastically and will start to yield and so irreversibly consolidate.

This compressive yield stress $P_y(\phi)$ is an implicit function of the strength of the interparticle bridging forces and possibly the previous shear history of the system, which will determine the primary floc size and internal structure. Buscall and White (1987) describe the estimation of the yield stress $P_y(\phi)$ from a batch centifuge experiment. A sample of suspension is spun at a particular speed until an equilibrium height h_{eq} is reached. The speed is increased and the exercise is repeated. The data comprise a curve of h_{eq} vs. g, the acceleration at the bottom of the tube. Then, the equations governing the network can be shown to yield a parametrization in g of the volume fraction and the solids stress at the bottom of the tube. The more complex expressions can be simplified giving a good initial estimate:

$$P(0)=P_y[\phi(0)]\approx\Delta\rho g\phi_0 h_0\left(1-\frac{h_{eq}}{2R}\right) \tag{2a}$$

$$\phi(0)=\frac{\phi_0 h_0\left[1-\frac{1}{2R}\left(h_{eq}+g\frac{dh_{eq}}{dg}\right)\right]}{\left[\left(h_{eq}+g\frac{d(gh_{eq})}{dg}\right)\left(1-\frac{h_{eq}}{R}\right)+\frac{h_{eq}^2}{2R}\right]} \tag{2b}$$

Thus, Eqs. 2a and 2b evaluated at each g value determine $P_y(\phi)$ parametric in g. This method has been used by Buscall et al. (1982) and De Guingand (1986) on experimental data. The resulting yield stress estimates were fitted with power law curves of the type:

$$P_y(\phi)=k\left[\left(\frac{\phi}{\phi_g}\right)^n-1\right] \tag{3a}$$

$$P_y(\phi)=k\left[\frac{\phi}{\phi_g}-1\right]^m \tag{3b}$$

with various values of n or m. Here ϕ_g is called the gel point and is the value below which $P_y(\phi)$ cannot be experimentally distinguished from zero. It may be considered the volume fraction at which all the primary flocs become interconnected. This concept of a network pressure is used later in the kinetic description of consolidation.

The dynamics of the one-dimensional model for cylindrical filter presses, where the suspended material is compressed by a piston at one face and only the liquid passes through a porous membrane at the other face, was studied by Landman et al. (1991). Closely related to such a filter is a vacuum filter, where pure liquid such as the drained liquid, is continuously fed into the top of the cylinder. The initial amount of solid particles remains constant throughout operation. When the filter is run at steady state, it provides a convenient and easy method for measuring the hindered settling factor $r(\phi)$. It is similar to the method of calculating the $P_y(\phi)$ proposed by Buscall and White (1987), as described briefly above, in that the flow rate through the filter (rather than the gravitational strength) is used to parameterize the volume fraction at the membrane and the quantity to be measured, $r(\phi)$. Unfortunately, at this stage there are no experimental data available so that we must apply our methods to numerically simulated data. The results here, however, are very promising and we are hopeful that this will encourage an experimental program. The theoretical work of Buscall and White (1987) initiated a new experimental technique for calculating $P_y(\phi)$ (Buscall, 1990; Buscall et al., 1988; De Guingand, 1986). We hope that this study will stimulate an experimental program in a similar way.

One-Dimensional Model for a Vacuum Filter

Consider a cylinder containing the suspension subject to a pressure difference across the top and bottom face. Initially the container is filled with a flocculated suspension. Only the liquid is allowed to pass through a porous membrane at the bottom face, while again only liquid is added to the top of the container, as in Figure 1. There are essentially two modes of operation of such a filter: (1) the rate of liquid expression is a specified function of time; (2) the applied pressure difference is a specified function of time.

Usually plug flow is assumed in these types of problems. In fact, the assumption that the volume fraction ϕ does not vary much over a cross-section of the cylinder leads to similar equations in the cross-sectionally averaged velocities and pressure. The equations required to describe the dynamics are a force balance equation on the particles and one on the suspending liquid, a mass conservation equation on the particles and on the liquid, and a kinetic equation for the particles.

In the force balance equations, essentially the hydrodynamic drag forces will balance the pressure gradients, assuming that gravitational forces are small compared to the applied pressure difference. This first model assumes, as usual, that the inertial

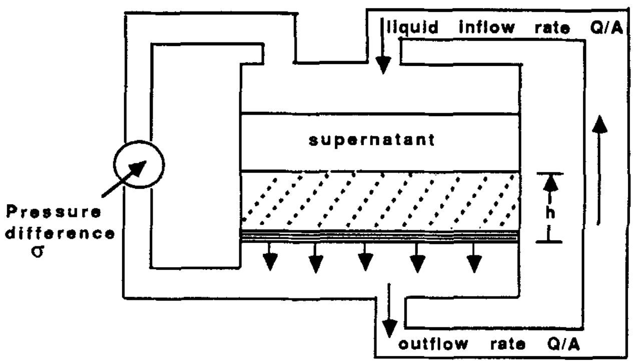

Figure 1. Cylindrical container used in vacuum filtration.

terms and the shear forces in the bulk of the fluid and those exerted by the container walls are smaller than the other terms. For flocculated systems, experimental observation (Buscall et al., 1982) has shown that the shear yield stresses are smaller than the compressive stresses. With these assumptions, the particle and fluid force balance equations, respectively (Buscall and White, 1987; Landman Sirakoff and White, 1991; Tiller, 1981), are:

$$-\frac{\lambda\eta a_p}{V_p}\phi r(\phi)(\mathbf{u}-\mathbf{w})-\frac{\partial p_s}{\partial z}\hat{\mathbf{z}}=0 \tag{4}$$

$$-\frac{\lambda\eta a_p}{V_p}\phi r(\phi)(\mathbf{w}-\mathbf{u})-\frac{\partial p_l}{\partial z}\hat{\mathbf{z}}=0 \tag{5}$$

The first term in Eq. 4 is the hydrodynamic drag exerted by the suspending fluid on the flocculated particles (where η is the fluid viscosity, a_p is the particle size, which for spheres is the radius, V_p is the particle volume, and λ is the Stokes drag coefficient, which for spheres is 6π). As discussed above, the hindered settling factor $r(\phi)$ accounts for the hydrodynamic interactions among the particles. The first term in Eq. 4 is the corresponding drag term exerted by the suspending liquid on the flocculated particles.

Conservation of particles and fluid masses requires the continuity equation:

$$\frac{\partial\phi}{\partial t}+\nabla\cdot(\phi\mathbf{u})=0 \tag{6}$$

$$\frac{\partial(1-\phi)}{\partial t}+\nabla\cdot[(1-\phi)\mathbf{w}]=0 \tag{7}$$

Buscall and White (1987) put forward the following constitutive equation as modeling the kinetics of the flocculated suspension:

$$\frac{D\phi}{Dt}=\begin{cases}0, & p_s<P_y(\phi) \quad (8a)\\ \kappa(\phi)[p_s-P_y(\phi)], & p_s\geq P_y(\phi) \quad (8b)\end{cases}$$

Here, D/Dt is the material derivative, $\kappa(\phi)$ is the dynamic compressibility of the suspension, and the compressive yield stress $P_y(\phi)$ is the value of the network pressure at which the network under compression will start to irreversibly consolidate.

Suppose that this filter operates under steady-state conditions: that is, we will ignore the transient effects. In this state, the filter cake is fully formed, ϕ remains the same for time t, and all the particles are at rest, so that $\mathbf{u}=0$. Also the flow rate of pure liquid at the feed point must exactly match the drainage rate through the porous membrane, so that the height of the mixture in the container remains fixed, as h_0, say. Writing

$$\mathbf{w}=-w\hat{\mathbf{z}} \tag{9}$$

the continuity equations (Eqs. 6-7) give:

$$(1-\phi)w=\frac{Q}{A} \tag{10}$$

where Q is the total fluid volumetric flow rate though the filter, with constant cross-sectional area A. Hence, the fluid velocity w can be eliminated from Eq. 4 to give:

$$\frac{dp_s}{dz}=-\frac{\lambda\eta a_p}{V_p}\frac{\phi r(\phi)}{1-\phi}\frac{Q}{A} \tag{11}$$

The kinetic equation (Eq. 8) reduces to ϕ remaining constant in the region where $p_s\leq P_y(\phi)$ (this is the clarification and nonconsolidation regions), while in the consolidation region Eq. 8b becomes:

$$p_s=P_y(\phi). \tag{12}$$

Finally, the force balance equations (Eqs. 4-5) can be combined to give the sum of the pressures being just the imposed pressure difference σ:

$$p_s+p_l=\sigma. \tag{13}$$

Let ϕ_0($\phi_0>\phi_g$, that is, the flocculated suspension is fully networked) be the initial volume fraction in the container. Then, $\phi_0 h_0 A$ is the total volume of solids. Clearly, the particle pressure must be zero at the top of the material (that is, the input point for the liquid):

$$p_s(h_0)=0. \tag{14}$$

The last boundary condition is related to the pressures evaluated at the membrane. Using Darcy's law for flow through the filter are $z=0$,

$$w(0)=\frac{k_f}{z}\frac{dp_l}{dz}(0)\sim\frac{k_f}{\eta l_f}[p_l(0)-p_{\text{atm}}]=\frac{k_f}{\eta l_f}p_l(0)$$

where k_f is the filter permeability which will be a function of $\phi(0)$ (in cases where clogging or fouling of the membrane occurs), l_f is the filter thickness and we have scaled the pressure so that p_{atm} is zero. Therefore, Eq. 13 gives:

$$p_s+\frac{\eta l_f}{k_f}\frac{Q}{A}\frac{1}{1-\phi}=\sigma \text{ at } z=0. \tag{15}$$

We now refer back to the two modes of operation for this vacuum filter:

1. When the fluid expression rate Q/A is specified, then p_s and ϕ can be solved uniquely, and Eqs. 15 and 13 determine the applied pressure difference and liquid pressure, respectively.
2. On the other hand, when the applied pressure difference σ is specified at all times, the fluid expression rate Q/A is determined at the same time as the solution to the system, using Eqs. 11 and 15.

However, for case 2, knowledge of the membrane permeability (possibly as a function of ϕ) is required for this case. Our interest will be on case 1, since the solution for p_s and ϕ are independent of the filter permeability. In fact, the experimental technique will be able to determine this variable as well as $r(\phi)$.

Before solving the equations in the whole container, some information regarding the zonal behavior—clarification, nonconsolidation, and consolidation zones—can be obtained. Since the pressure p_s is zero at the top h_0 and the consolidation process is irreversible by the kinetic equation (Eq. 8), if there is to be any consolidation there must be a region at the top of the material $h<z<h_0$ which is pure liquid with $\phi=0$. Then, since the solids pressure must be zero at the height where the bed starts, $p_s(h)=0$, there is a region $z_c<z<h$ where the volume fraction is given by $\phi=\phi_0$ by Eq. 8, and since the network pressure p_s in this region is not sufficiently large to exceed the yield stress: $p_s<P_y(\phi_0)$. In this region, Eq. 11 can be solved giving:

$$p_s=\frac{\lambda\eta a_p}{V_p}\frac{\phi_0 r(\phi_0)}{1-\phi_0}\frac{Q}{A}(h-z). \tag{16}$$

The position z_c, which marks the boundary between the unconsolidated and consolidated zone, is the point where the network pressure becomes equal to the compressive yield stress $P_y(\phi_0)$ of the network:

$$P_y(\phi_0)=\frac{\lambda\eta a_p}{V_p}\frac{\phi_0 r(\phi_0)}{1-\phi_0}\frac{Q}{A}(h-z_c). \tag{17}$$

Clearly, the existence of such a consolidation region puts a lower bound on the flow rates Q as:

$$P_y(\phi_0)<\frac{\lambda\eta a_p}{V_p}\frac{\phi_0 r(\phi_0)}{1-\phi_0}\frac{Q}{A}h_0. \tag{18}$$

To obtain the solids pressure and the volume fraction in the whole container, the nonlinear differential equation (Eq. 11) must be solved. It is convenient to rescale some of the variables and introduce some parameters as:

$$Z=\frac{z}{h_0},\ H=\frac{h}{h_0},\ Z_c=\frac{z_c}{h_0} \tag{19a}$$

$$\Pi(Z)=\frac{p_s(z)}{P_y(\phi_0)} \tag{19b}$$

$$f_y(\phi)=\frac{P_y(\phi)}{P_y(\phi_0)} \tag{19c}$$

$$R(\phi)=\gamma r(\phi) \tag{19d}$$

$$\Sigma=\frac{\sigma}{P_y(\phi_0)} \tag{19e}$$

$$\gamma=\frac{\lambda\eta a_p}{V_p} \tag{19f}$$

$$\beta=\frac{Q}{A}\frac{h_0}{P_y(\phi_0)} \tag{19g}$$

$$K=\frac{\eta l_f}{k_f}\frac{Q}{A}\frac{1}{P_y(\phi_0)}. \tag{19h}$$

Here, $R(\phi)$ combines the two variables γ and $r(\phi)$, which will be determined (essentially the γ appears in u_0 in Eq. 1). These always appear together in a force balance equation such as Eqs. 4–5.

Then, the differential equation for the dimensionless solids pressure is:

$$\frac{d\Pi}{dZ}=-\beta\frac{\phi R(\phi)}{(1-\phi)}, \tag{20}$$

with conditions:

$$\Pi=f_y(\phi)\ \text{if}\ \Pi>1 \tag{21a}$$

$$\Pi(H)=0,\ \Pi(Z_c)=1 \tag{21b}$$

$$\phi=\phi_0\quad Z_c<Z<H \tag{21c}$$

$$\int_0^H \phi dz=\phi_0 \tag{21d}$$

$$\Pi(0)-K\frac{1}{(1-\phi(0))}=\Sigma. \tag{21e}$$

Estimation of the $R(\phi)$ from Experiments

The vacuum filter is run at steady state for a volume fraction ϕ_0 and initial height h_0, so that the volume of solids is $\phi_0 h_0 A$ for a certain value of liquid flux Q/A. The bed height h, which separates the region of clarification with the suspension region, and the pressure difference, required to sustain this flux σ, are measured. Then, the flow rate is increased, so that this experiment is repeated a number of times for several values of Q/A. Then, the basic data comprise points h and σ vs. Q/A. Since $P_y(\phi_0)$ has already been calculated, the data for H and Σ vs. β are known. At this stage, we can think of these as defining curves $H(\beta)$ and $\Sigma(\beta)$.

Given the data, the object is to calculate $R(\phi)$. We know from Eq. 20 that:

$$\int_0^{\Pi(Z)}\frac{(1-\phi)}{\phi R(\phi)}d\Pi=-\beta\int_H^Z dZ=\beta(H-Z). \tag{22}$$

Here, $R(\phi)=R[\phi(\Pi)]$, and $\phi(\Pi)$ is taken as the function:

$$\phi(\Pi)=\begin{cases}\phi_0 & \Pi\le 1\\ \Phi_y & \Pi>f_y(\phi)\end{cases}$$

where $\Phi_y(\Pi)$ is the inverse to $f_y(\phi)$:

$$\Phi_y[f_y(\phi)]=\phi.$$

In particular, at $Z=0$, Eq. 22 becomes:

$$\int_0^{\Pi(0)}\frac{(1-\phi)}{\phi R(\phi)}d\Pi=\beta H. \tag{23}$$

Differentiating Eq. 23 with respect to β we obtain:

$$\frac{1-\phi(0)}{\phi(0)R[\phi(0)]}\frac{d\Pi(0)}{d\beta}=\frac{d(\beta H)}{d\beta}. \tag{24}$$

We also have the conservation of total mass equation (Eq. 21d) which can be rewritten as:

$$\phi_0 = \int_0^H \phi dz = \int_{\Pi(0)}^0 \phi \frac{dZ}{d\Pi} d\Pi;$$

and using Eq. 20:

$$\int_0^{\Pi(0)} \frac{(1-\phi)}{R(\phi)} d\Pi = \beta\phi_0$$

where again ϕ is thought of as a function of Π. Then, differentiation with respect to β gives

$$\frac{1-\phi(0)}{R[\phi(0)]} \frac{d\Pi(0)}{d\beta} = \phi_0. \tag{25}$$

Substitution of Eq. 25 into Eq. 24 calculates $\phi(0)$ as:

$$\phi(0) = \frac{\phi_0}{\dfrac{d(\beta H)}{d\beta}} \tag{26}$$

This gives the result that differentiation of the $H(\beta)$ data will determine the values of $\phi(0)$ as a function of β. The corresponding solids pressure is then evaluated using:

$$\Pi(0) = f_y[\phi(0)] \tag{27}$$

for each value of β. Differentiation of the constructed $\Pi(0)$ data with respect to β calculates $R[\phi(0)]$ according to Eq. 25. Hence, this procedure gives $\phi(0)$ and $R[\phi(0)]$ parameterized in β, and hence the function $R(\phi)$ can be determined.

The filter permeability as expressed in K from Eq. 19h can be determined using the pressure difference data $\Sigma(\beta)$ as:

$$\Pi(0) - \frac{K}{1-\phi(0)} = \Sigma \tag{28}$$

Since the permeability will depend on the concentration of the solids $\phi(0)$ at the membrane, this gives a β parameterization of K, and hence K as a function of ϕ.

Equation 18 determines the smallest value of the flow rate, and hence a corresponding $\beta = \beta_0$ required for consolidation to occur for the height $H<1$. Here,

$$\beta_0 = \frac{1-\phi_0}{\phi_0 R(\phi_0)} \tag{29}$$

when $\beta = \beta_0$, $H(\beta_0) = 1$, and $\phi(0) = \phi_0$. Therefore, since β_0 can be calculated experimentally, $R(\phi_0)$ is known. This will help the numerical differentiation procedure, acting like a constraint.

In this procedure, it is assumed that the function $f_y(\phi)$, and hence $P_y(\phi)$, is known for the required range of values of the solids volume fraction, which develop in the filtration experiment (because we need Eq. 27 to construct the pressure data). This would be determined if, for example, a batch centrifuge experiment as described by Buscall and White (1987) had been performed for the flocculated suspension under discussion. If, however, one was only working with the flocculated suspension in a pressure or vacuum filter, rather than gravity or centrifuge thickening, then a function containing the hindered settling function R can be recovered using the same experimental procedure just discussed. To show this, we need to rewrite only the pressure equation (Eq. 20) in terms of the volume fraction using Eq. 21a. This gives:

$$D(\phi)\frac{d\phi}{dZ} = -\beta \tag{30}$$

where

$$D(\phi) = \frac{f_y'(\phi)(1-\phi)}{\phi R(\phi)} \tag{31}$$

For the study of time-dependent pressure and vacuum filtration (Landman et al., 1991), the function $\phi D(\phi)$ acts like a diffusion coefficient for the transport of the solids. Hence, if only this type of dewatering is of interest, then it is not necessary to know $R(\phi)$ but the combination function $D(\phi)$. The analysis of the $H(\beta)$ data proceeds in the same way as above, where now Π is replaced by $f_y(\phi)$. Again $\phi(0)$ as a function of β is determined by Eq. 26, while Eq. 25 is replaced by:

$$\phi(0)D[\phi(0)] \frac{d\phi(0)}{d\beta} = \phi_0. \tag{32}$$

Here, now $\phi(0)$ is differentiated and substituted into Eq. 32, which determines the necessary parameterization of D with respect to β.

From Eq. 29, $R(\phi_0)$ is known. To know the initial value of $D(\phi_0)$ at this stage (that is, before the differentiation procedure), the slope $f_y'(\phi_0)$ must be known. However, for the pressure filtration formulation, $P_y(\phi)$ does not need to be predetermined for all the required range of values of ϕ.

Numerical Simulation

The procedure to recover either $R(\phi)$ or $D(\phi)$ involves numerical differentiation of the data H vs. β and then the further differentiation of $\Pi(0)$ or $\phi(0)$ vs. β; this is effectively two derivatives of the data. The problem of estimating derivatives of nonexact data (data corrupted with noise) is a notoriously difficult problem (Anderssen and Bloomfield, 1972; Davies and Anderssen, 1986a,b). Since numerical differentiation is an unstable or ill-posed problem, it is necessary to introduce a stabilization technique into the differentiation method. (Just taking divided differences is satisfactory if the data are noise-free; with any amount of noise, it is totally inadequate.) For the problem here, we used a Fourier differentiation and smoothing technique, called A_0 optimality (Davies, 1991; Hassan, 1989), which is an improvement of the cross-validation technique of Davies and Anderssen (1986a). Some of the pertinent components in implementing the technique for these particular data are explained in the Appendix. As far as the experimenter is concerned, the measurements for the bed height should be taken at equally spaced flow rates. Suppose that $N+1$ data values are made at β_n, $n = 0, 1, \ldots, N$, where the first one β_0 is the critical flow rate defined in Eq. 29. N should be

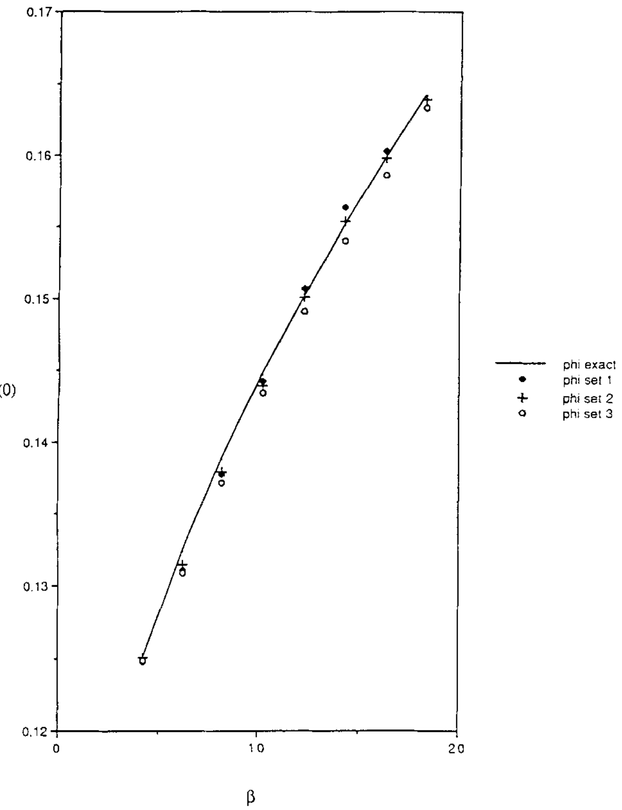

Figure 2a. $\phi(0)$ vs. β for $N=10$, $n=5$ in Eq. 3a for three data sets with 1% error.

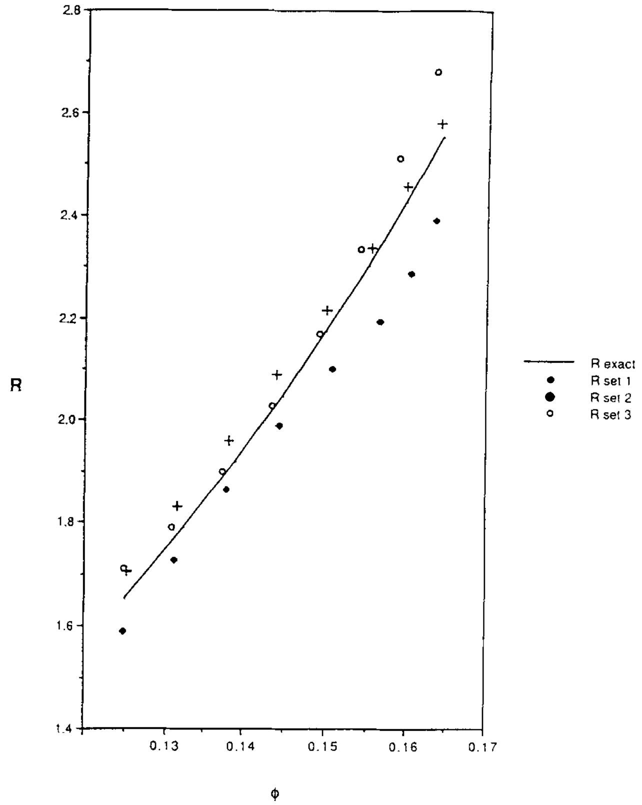

Figure 2b. R vs. ϕ for $N=10$, $n=5$ in Eq. 3a for three data sets with 1% error.

an even number. Then, the differentiation technique evaluates $\phi(0)$ at β_n, $n=0, 1, \ldots, N-1$, and the R or D at β_n, $n=0, 1, \ldots, N-3$. Hence, on eliminating the parameterization $R(\phi)$ and $D(\phi)$ will be known at N-2 points ϕ.

In the following examples, the data are generated using $\phi_0=0.125$ and $\phi_g=0.1$ and the yield stress at $P_y(\phi_0)$ has the form of Eq. 3a, with either $n=5$ or $n=3$ [in order to see different behavior in $D(\phi)$]. Various forms for $r(\phi)$ and values of γ were investigated. Here, we will concentrate on:

$$r(\phi)=\left(1-\frac{\phi}{0.64}\right)^{-5.5}, \gamma=0.5 \tag{33}$$

where $0.64=\phi_{cp}$. Normally distributed noise, with zero mean variance σ^2, was added to the true value of $H(\beta)$. This has to be done in a way that reflects any real experiment. Since for any experiment the flow rate is progressively increased in fixed steps, the height of the bed will decrease with increasing flow rate; if a measurement did not reflect this monotonicity, the data would not be acceptable by an experimenter. Hence, given a spacing in β, the noise level is chosen so that the data are monotonically decreasing. Several runs were done with the same true data and fixed noise level to test the reproducibility of the results. This corresponds to different simulated experiments. In the figures they are denoted as set 1, set 2, and so on. (It is assumed here that the random error in β is small compared to the error in the bed height measurements.)

In Figures 2a and 2b, the number of data points $(N+1)$ is 11, so that the $R(\phi)$ plot has eight points determined from the Fourier technique. We have simulated three experimental runs,

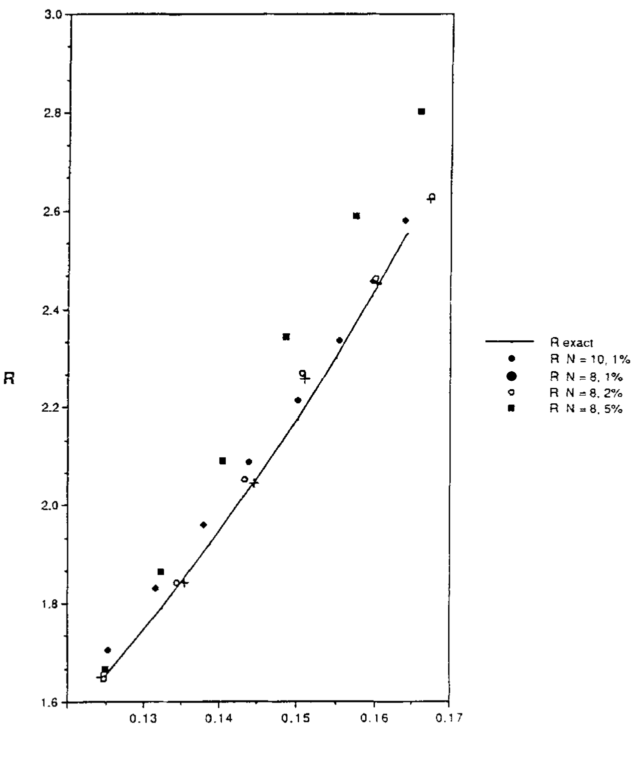

Figure 3. R vs. ϕ for $n=5$ in Eq. 3a, $N=10$ and 1% error and $N=8$ with 1%, 2% and 5% error.

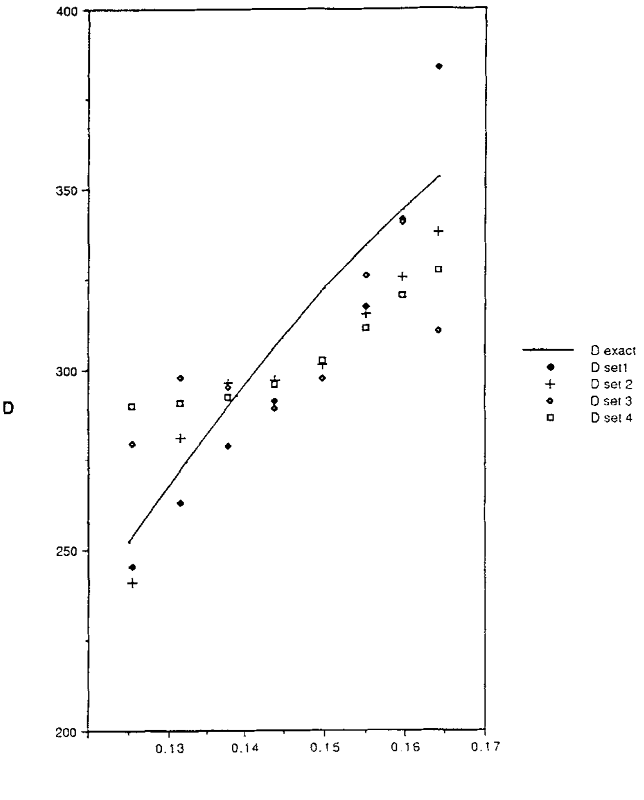

Figure 4. *D* vs. ϕ for $n=5$ in Eq. 3a, $N=10$ and 1% error.

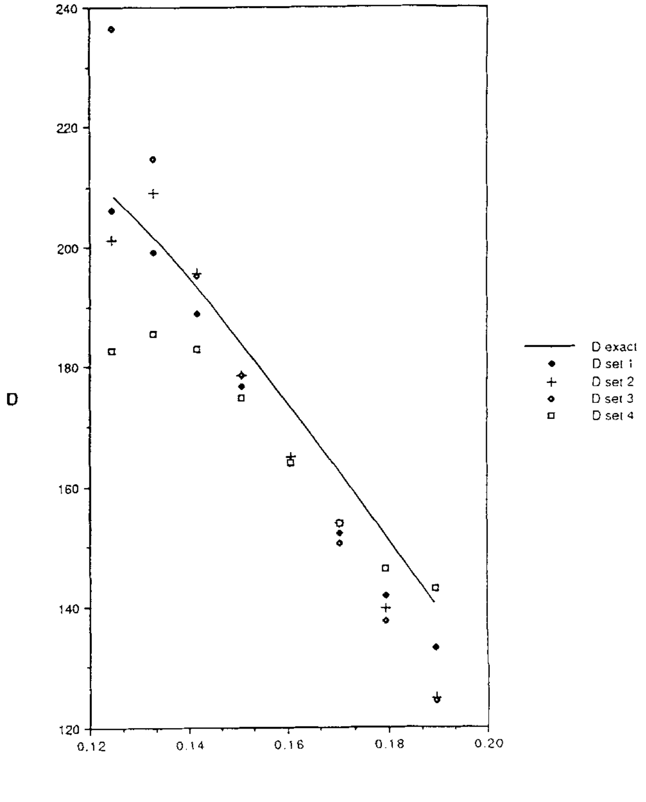

Figure 5. *D* vs. ϕ for $n=3$ in Eq. 3a, $N=10$ and 1% error.

with 1% noise. Figure 2a shows that $\phi(0)$ is reconstructed very well. There is more error in Figure 2b, because it is essentially derived by taking the derivative of the data in Figure 2a. In Figure 3 the effect of decreasing the number of data points ($N=8$) over the same range is examined. The noise levels are increased from 1% to 5%. As the noise level increases, the major error appears at the larger volume fraction levels because we use the fact that $R(\phi_0)$ is known as a constraint (see the Appendix).

We examined the effect on the recovered data when changing the power in $r(\phi)$, for example, increasing the power -5.5 to -4.5 in Eq. 33. The solutions obtained had similar clustering about the true value; since decreasing the power changes the values of $R(\phi)$ by approximately 25%, the techniques employed were able to distinguish between the two forms of $R(\phi)$. Other forms of $r(\phi)$ were also investigated, and this verified that the method is fairly insensitive to the form of this function.

Figures 4 and 5 show the D vs. ϕ, using $r(\phi)$ and γ from Eq. 33; the only difference between the two is that the power in the yield stress in Eq. 3a is $n=5$ and $n=3$ in Figures 4 and 5, respectively. They all have identical noisy data, with 1% noise level. There is much more scatter in these figures for one reason in particular. As stated previously, the only information required to solve for D is knowledge of $P_y(\phi_0)$, where for $R(\phi)$ one needs $P_y(\phi)$ over the range of interest of ϕ. However, if the derivative $P_y'(\phi_0)$ is known (and experimentally this would entail knowing $P_y(\phi)$ over a small range of ϕ near ϕ_0), then D at β_0 is known *a priori* and can be used as a constraint in the numerical differentiation regularization. In this case, sets 3 and 4 would be discarded in favor of set 1 or 2 in Figures 4 and 5. This method for recovering D is limited, but is still a way of getting some information regarding the relevant rheological properties for flocculated suspensions. If $P_y(\phi)$ is known by some other means, then it is preferable to recover $R(\phi)$ rather than $D(\phi)$ because the monotonicity constraint on R can be used to eliminate the more scattered recovered data.

It should be noted that if the membrane resistance (included in the parameter K) is negligible, then Eq. 28 still gives another piece of information. Instead of using Eq. 27 and essentially twice differentiating the $H(\beta)$ data, the $\Sigma(\beta)$ can be differentiated once to give $[d\Pi(0)]/d\beta$, which simplifies Eq. 25 to:

$$\frac{1-\phi(0)}{R(\phi(0))}\frac{d\Sigma}{d\beta}=\phi_0. \tag{34}$$

Now, Eqs. 26 and 34 can be used as parametrizations in β to recover $R(\phi)$.

These results show a good recovery of the hindered settling factor or the "diffusion" coefficient. We are hopeful that this will encourage an experimental program as did earlier comparable work on the recovery of the compressive yield stress.

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Notation

a_p = particle dimension
A = constant cross-sectional area of the vacuum filter
$D(\phi)$ = diffusion coefficient combining the hindered settling factor and the yield stress
$f_y(\phi)$ = scaled yield stress function
g = acceleration due to centrifuge experiment of Buscall and White (1987)
h_0 = fixed height of the mixture
h = height of the bed
h_{eq} = height of the bed in the centrifuge experiment of Buscall and White (1987)
H = scaled height of the bed
k = constant in the yield stress function
k_f = membrane permeability
l_f = membrane thickness
m = index in yield stress function, Eq. 3b
n = index in yield stress function, Eq. 3a
p_l = liquid pressure
p_s = solids pressure
$P_y(\phi)$ = compressive yield stress of flocculated suspension
Q = total fluid volumetric flow rate
R = radial distance from the centrifuge center to the bottom of the sedimentation tube
$r(\phi)$ = hydrodynamic hindered settling factor
$R(\phi)$ = hydrodynamic hindered settling factor, combined with shape factor γ
t = time
u_0 = Stokes settling velocity of an isolated particle
$\boldsymbol{u}$ = solids velocity vector
V_p = average particle volume
w = magnitude of fluid velocity vector
$\boldsymbol{w}$ = fluid velocity vector
z = vertical spatial coordinate
z_c = height of boundary between the unconsolidated and consolidated zone
Z = scaled vertical spatial coordinate
Z_c = scaled height z_c

Greek letters

β = scaled liquid flow rate
$\Delta\rho$ = density difference between particle and fluid phases
γ = shape factor
ϕ = volume fraction of suspension occupied by solids
ϕ_{cp} = close packing volume fraction
ϕ_g = gel point of a flocculated suspension
ϕ_0 = initial volume fraction
$\Phi_y(\Pi)$ = inverse function of $f_y(\phi)$
η = fluid viscosity
λ = Stokes drag coefficient ($=6\pi$)
$\kappa(\phi)$ = dynamic compressibility of the flocculated network
K = scaled membrane permeability
$\Pi(Z)$ = scaled network pressure
σ = pressure difference
Σ = scaled pressure difference

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Appendix

Discussed in this section are some of the details of the Fourier differentiation method for evaluating one derivative of nonexact data applied to this particular problem.

1. The technique requires $N+1$ data points y_n taken at equally spaced points $x_n = n/N$ $(n=0,1,2,\ldots,N)$. Hence,

$$x_n = \frac{\beta_n - \beta_0}{\beta_N - \beta_0} \tag{A1}$$

when N is an even number. Here,

$$y_n = g(x_n) + \epsilon_n \tag{A2}$$

where ϵ_n is white noise.

2. After suitable preprocessing of the data y_n (discussed below), the method calculates the Fourier transform of the data and then multiplies this vector by $i\omega$, which corresponds to taking the first derivative in x space. To stabilize this process, the Fourier transform is multiplied by a suitable regularizing filter:

$$Z(\lambda,p) = \frac{1}{1+\lambda\omega^{2p+2}}, \tag{A3}$$

where the nonnegative integer p is the order of regularization, and λ is a parameter chosen in a suitable way (for example, minimizing a function related to the difference between the true and numerical derivative $U(\lambda)$(Davies, 1991; Hassan, 1989).

3. Finally, the inverse Fourier transform of the resulting vector is evaluated and the solution is suitably post-processed (determined exactly by the preprocessing before introducing the frequency space).

4. The preprocess step is extremely important. This step simulates the underlying data function. If it is not carried out effectively, the solution suffers high-frequency distortion due to this lack of proper processing rather than due to any noise in the experimental data. Preprocessing relies on the value of

$\Delta_1 = g(0) - g(1)$. If the order of regularization $p \geq 1$, then the higher-order differential differences $\Delta_{v+1} = g^{(v)}(0) - g^{(v)}(1)$, $v = 1,\ldots,p$, are also required. A suitable polynomial $f(x)$ of the order $p+1$ for the data can be constructed so that the new processed datum, $G(x_n) = G_n = y_n + f(x_n)$, obeys periodic boundary conditions at x equals zero and one $[G^{(v)}(0) = G^{(v)}(1)$ for $v = 0,\ldots,p]$, as well as the sum of all the G_n points being zero.

Now consider how this procedure works for our problem. When calculating the first derivative of βH, the values of these functions are known exactly at β_0 (using Eq. 26 and the definition of β_0), but the value at the other end β_N is not known exactly but only with a certain error ϵ_N. However, the best estimate of Δ_1 should be made, and this is the difference between the exact value at β_0 and the (nonexact) value at β_N. Different values of the regularization parameter were investigated. For $p = 0$, it was found that the recovery of $R(\phi)$ was poor, while recovery using $p = 1$ was good when Δ_2 was chosen suitably.

When calculating

$$\frac{d(\beta H)}{d\beta} \text{ with } p = 1,$$

then

$$\Delta_2 = \left.\frac{d(\beta H)}{d\beta}\right|_{\beta_0} - \left.\frac{d(\beta H)}{d\beta}\right|_{\beta_N}$$

is required. The first term equals unity (from Eq. 26), but the second term is completely unknown. Now we are faced with the question of how to choose Δ_2 for the preprocessing required for $p = 1$. Fortunately, there is *a priori* information we can use, namely that $\phi(0)$ is a monotonically increasing function of β and $\phi(0)$ at β_0 should be close to ϕ_0. We found that these "constraints" were enough to choose Δ_2 to obtain good results for $\phi(0)$. The value of the regularization parameter λ was chosen to minimize a certain function $U(\lambda)$ (Davies, 1991; Hassan, 1989).

Similar problems occur for the choice of Δ_1 and Δ_2 for the derivative calculations for $\Pi(0)$ or $\phi(0)$. These are known at β_0 (and equal unity and ϕ_0, respectively, from the definition of β_0). The way outlined above gives the best estimate for Δ_1, while various guesses of Δ_2 (for example, negative Δ_2's for the Π calculation since R is monotonic in β) and choice of the regularization parameter λ can be attempted so that the resulting R is a monotonic function of β and $R(\beta_0)$ is close to $R(\phi_0)$, as given in Eq. 29. For the D calculation, D need not be monotonic. Also $D(\beta_0)$ is only known if $f_y'(\phi_0)$ [that is, $P_y'(\phi_0)$] has been determined. Since there are fewer constraints on this calculation, there is more variation in the possible outcomes of the numerical differentiation, as explained in the main text.

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